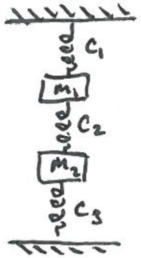


Return to springs and masses moving in one dimension. Allow displacement to change over time.

$$\begin{Bmatrix} u_1 = u_1(t) \\ u_2 = u_2(t) \end{Bmatrix} \to \begin{Bmatrix} e_1 = e_1(t) \\ e_2 = e_2(t) \\ e_3 = e_3(t) \end{Bmatrix} \to \begin{Bmatrix} f_1 = f_1(t) \\ f_2 = f_2(t) \end{Bmatrix}$$

The force balance equation

$$Ku = f$$

must be modified to include <u>acceleration</u>:

$$\underbrace{Mu''}_{\substack{\text{"Mass matrix"} \\ \text{Diagonal matrix of masses.}}} + Ku = f$$

$$\begin{bmatrix} m_1 u_1'' \\ m_2 u_2'' \\ \vdots \end{bmatrix} = \begin{bmatrix} m_1 & 0 & \cdots \\ 0 & m_2 & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} u_1'' \\ u_2'' \\ \vdots \end{bmatrix}$$

We will focus on isolated (i.e. "free") systems where <u>all</u> internal <u>force</u> from stretching/contracting springs is converted to <u>acceleration</u> of masses:

$$Mu'' + Ku = 0$$

Recall: Everything is a function of time, so $u'' = \frac{d^2}{dt^2}u(t)$. The force balance equation

$$Mu'' + Ku = 0$$

is a <u>2nd order differential equation</u>.

"Fancy Differential Equations Stuff" is <u>not</u> necessary to solve $Mu'' + Ku = 0$.

→ These are masses on springs... they should oscillate with some amplitude and frequency

<u>Try to solve for amplitude & frequency</u>:

$$\begin{Bmatrix} u_1(t) = a_1 \cos(\omega t) \\ u_2(t) = a_2 \cos(\omega t) \end{Bmatrix} \to u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos(\omega t)$$

a_1 = amplitude for mass 1
a_2 = amplitude for mass 2
ω = frequency

$$\begin{Bmatrix} \frac{d}{dt} u_1(t) = -a_1 \omega \sin(\omega t) \\ \frac{d}{dt} u_2(t) = -a_2 \omega \sin(\omega t) \end{Bmatrix} \to u' = -\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \omega \sin(\omega t)$$

$$\cdots \to u'' = -\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \omega^2 \cos(\omega t)$$

(solving for amplitude & frequency ...)

→ Plug u & u'' into $Mu'' + Ku = 0$

$$M\left(-\begin{bmatrix}a_1\\a_2\end{bmatrix}\omega^2\cos(\omega t)\right) + K\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}\cos(\omega t)\right) = 0$$

$$K\begin{bmatrix}a_1\\a_2\end{bmatrix}\cancel{\cos(\omega t)} = M\begin{bmatrix}a_1\\a_2\end{bmatrix}\omega^2\cancel{\cos(\omega t)}$$

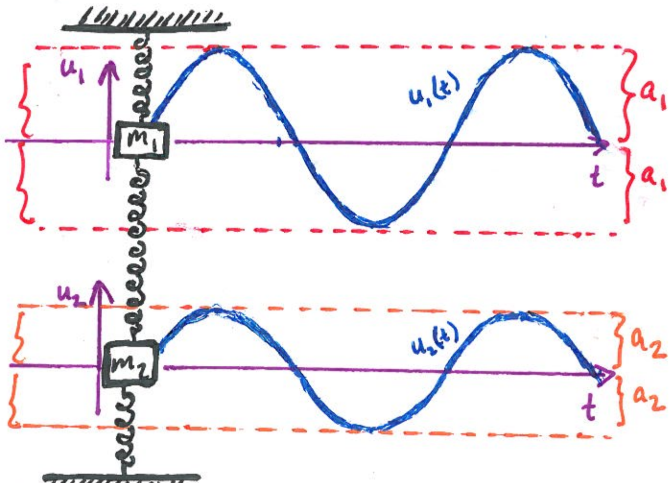
These cancel because we can choose t so that $\cos(\omega t) \neq 0$

$$M^{-1}K\begin{bmatrix}a_1\\a_2\end{bmatrix} = \omega^2\begin{bmatrix}a_1\\a_2\end{bmatrix}$$

$$\begin{bmatrix}1/m_1 & 0 & \cdots\\ 0 & 1/m_2 & \ddots\\ \vdots & \ddots & \ddots\end{bmatrix}$$ Diagonal matrix of 1/mass

(Note: $M^{-1}K$ divides each row of K by mass)

- Amplitudes are eigenvectors!
- Frequencies are sqrt of eigenvalues!



Amplitude of u_1 is a_1

Amplitude of u_2 is a_2

Both oscillate with frequency ω.

Note: Each eigenvalue & eigenvector gives two solutions

$$\underbrace{\begin{bmatrix}a_1\\a_2\end{bmatrix}\cos(\omega t)} \quad \text{and} \quad \underbrace{\begin{bmatrix}a_1\\a_2\end{bmatrix}\sin(\omega t)}$$

Solution with
initial position $= \begin{bmatrix}a_1\\a_2\end{bmatrix}$
initial velocity $= \begin{bmatrix}0\\0\end{bmatrix}$

Solution with
initial position $= \begin{bmatrix}0\\0\end{bmatrix}$
initial velocity $= \begin{bmatrix}\omega a_1\\\omega a_2\end{bmatrix}$

In general, solutions are a sum of these

$$c\begin{bmatrix}a_1\\a_2\end{bmatrix}\cos(\omega t) + d\begin{bmatrix}a_1\\a_2\end{bmatrix}\sin(\omega t)$$

→ the cos term sets initial position
the sin term sets initial velocity

Solutions where the masses all move together with the same frequency (synchronized) are called "Fundamental Modes of Oscillation"

→ Each eigenvalue & eigenvector of $M^{-1}K$ gives one Fundamental Mode of system

→ Most solutions are sums of multiple Fundamental Modes

Summary: Displacement functions in 1-dimensional oscillating spring systems are determined by eigenvalues & eigenvectors of $M^{-1}K$

→ eigenvalue gives frequency of fund. mode

$$\omega = \sqrt{\lambda}$$

→ eigenvector gives (relative) amplitudes

$$\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

Solution is

$$u(t) = \boxed{c_1 \mathbf{v}_1 \cos(\omega_1 t) + d_1 \mathbf{v}_1 \sin(\omega_1 t)}$$ Fundamental Mode #1

$$+ \boxed{c_2 \mathbf{v}_2 \cos(\omega_2 t) + d_2 \mathbf{v}_2 \sin(\omega_2 t)}$$ Fundamental Mode #2

$+ \cdots$ (other eigenvalue terms) $\cdots$ Other Fund. Modes

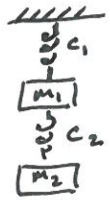
Determines initial position $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots$

Determines initial velocity $\omega_1 d_1 \mathbf{v}_1 + \omega_2 d_2 \mathbf{v}_2 + \cdots$

An Initial Value Problem (IVP) includes values for $u(0)$ and $u'(0)$, which can be used to solve for c_1, c_2, d_1, d_2, etc.

Remember K is stiffness matrix from before. $M^{-1}K$ divides each row by its mass.

EX: Find the fundamental modes for the following



$c_1 = 4$

$m_1 = 1/2$

$c_2 = 2$

$m_2 = 1/4$

First we should compute K and $M^{-1}K$

$$K = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}$$

$$M^{-1}K = \begin{bmatrix} 1/\tfrac{1}{2} & 0 \\ 0 & 1/\tfrac{1}{4} \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -4 \\ -8 & 8 \end{bmatrix}$$

Now we find eigenvalues & eigenvectors of $M^{-1}K$

$$\det \begin{bmatrix} 12-\lambda & -4 \\ -8 & 8-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 20\lambda + 64 = 0$$

$$(\lambda - 4)(\lambda - 16) = 0$$

$$\underline{\lambda = 4,\ 16} \rightsquigarrow \underline{\omega = 2,\ 4}$$

(eigenvectors on next page)

$\lambda = 4$ $(\omega = 2)$

$$\begin{bmatrix} 12-4 & -4 \\ -8 & 8-4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

use U from LU Decomp.

$$\begin{bmatrix} -8 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} a_2 &= \text{free} \\ a_1 &= -a_2/2 \end{aligned}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} a_2 \xrightarrow{a_2 = 2} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\lambda = 16$ $(\omega = 4)$

$$\begin{bmatrix} 12-16 & -4 \\ -8 & 8-16 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

use U from LU Decomp.

$$\begin{bmatrix} -4 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} a_2 &= \text{free} \\ a_1 &= -a_2 \end{aligned}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} a_2 \xrightarrow{a_2 = 1} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

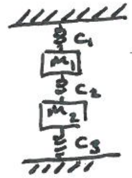
Fundamental Modes:

$$c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cos(2t) + d_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \sin(2t)$$

and

$$c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos(4t) + d_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin(4t)$$

EX: Give the general solution for the following:



$c_1 = 4$ $\quad m_1 = 1$

$c_2 = 1$ $\quad m_2 = 1/4$

$c_3 = 1$

$$K = \begin{bmatrix} 4+1 & -1 \\ -1 & 1+1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \qquad M^{-1}K = \begin{bmatrix} 5 & -1 \\ -4 & 8 \end{bmatrix}$$

$$\det \begin{bmatrix} 5-\lambda & -1 \\ -4 & 8-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 13\lambda + 36 = 0$$

$$(\lambda - 4)(\lambda - 9) = 0 \qquad \lambda = 4,\ 9$$

$\lambda = 4$, $\omega = 2$

$$\begin{bmatrix} 5-4 & -1 \\ -4 & 8-4 \end{bmatrix} = L \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$\hookrightarrow$ eigenv $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 9$, $\omega = 3$

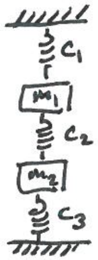
$$\begin{bmatrix} 5-9 & -1 \\ -4 & 8-9 \end{bmatrix} = L \begin{bmatrix} -4 & -1 \\ 0 & 0 \end{bmatrix}$$

$\hookrightarrow$ eigenv $= \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

General Solution:

$$u(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(2t) + d_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(2t) + c_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \cos(3t) + d_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \sin(3t)$$

EX: Solve the initial value problem.



$c_1 = 1 \qquad m_1 = 1 \qquad\qquad u(0) = \begin{bmatrix}2\\0\end{bmatrix}$

$c_2 = 4 \qquad\qquad$ with

$\qquad\qquad m_2 = 1 \qquad\qquad u'(0) = \begin{bmatrix}4\\-2\end{bmatrix}$

$c_3 = 1$

Write the answer as $u_1(t)$ & $u_2(t)$ = displacement of masses at time t

$$K = \begin{bmatrix}1+4 & -4\\-4 & 1+4\end{bmatrix} = \begin{bmatrix}5 & -4\\-4 & 5\end{bmatrix} \qquad M^{-1}K = \begin{bmatrix}5 & -4\\-4 & 5\end{bmatrix}$$

eigenval.

$$\det\begin{bmatrix}5-\lambda & -4\\-4 & 5-\lambda\end{bmatrix} = 0$$

$$\lambda^2 - 10\lambda + 9 = 0$$

$$(\lambda-9)(\lambda-1) = 0 \qquad \underline{\underline{\lambda = 1, 9}}$$

eigenvect.

$\underline{\underline{\lambda = 1}}$, $\omega = 1$: $\begin{bmatrix}5-1 & -4\\-4 & 5-1\end{bmatrix} = L\begin{bmatrix}4 & -4\\0 & 0\end{bmatrix}$ $\hookrightarrow$ eigenvect $= \begin{bmatrix}1\\1\end{bmatrix}$

$\underline{\underline{\lambda = 9}}$, $\omega = 3$: $\begin{bmatrix}5-9 & -4\\-4 & 5-9\end{bmatrix} = L\begin{bmatrix}-4 & -4\\0 & 0\end{bmatrix}$ $\hookrightarrow$ eigenvect $= \begin{bmatrix}-1\\1\end{bmatrix}$

general sol'n.

$$u = c_1\begin{bmatrix}1\\1\end{bmatrix}\cos(t) + d_1\begin{bmatrix}1\\1\end{bmatrix}\sin(t) + c_2\begin{bmatrix}-1\\1\end{bmatrix}\cos(3t) + d_2\begin{bmatrix}-1\\1\end{bmatrix}\sin(3t)$$

solve for c_1, c_2, d_1, d_2.

$\boxed{\text{Initial Position}}$

$$\begin{bmatrix}2\\0\end{bmatrix} = \underline{\underline{u(0)}} = c_1\begin{bmatrix}1\\1\end{bmatrix}\cancelto{1}{\cos 0} + \cancel{d_1\begin{bmatrix}1\\1\end{bmatrix}\sin 0} \to 0 + c_2\begin{bmatrix}-1\\1\end{bmatrix}\cancelto{1}{\cos 0} + \cancel{d_2\begin{bmatrix}-1\\1\end{bmatrix}\sin 0} \to 0$$

$$\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

Eigenvectors

2x2 inverse $\hookrightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix} = \frac{1}{1+1}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\end{bmatrix}$

$\boxed{\text{Initial Velocity}}$

$$\begin{bmatrix}4\\-2\end{bmatrix} = u'(0) = \cancel{c_1\begin{bmatrix}1\\1\end{bmatrix}\sin 0} \to 0 + d_1\begin{bmatrix}1\\1\end{bmatrix}1\cdot\cancelto{1}{\cos 0} + \cancel{c_2\begin{bmatrix}-1\\1\end{bmatrix}3\sin 0} \to 0 + d_2\begin{bmatrix}-1\\1\end{bmatrix}3\cdot\cancelto{1}{\cos 0}$$

$$\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1\cdot d_1\\3\cdot d_2\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix}$$

Eigenvectors

$\underline{\underline{\omega\cdot d}}$

$\hookrightarrow \begin{bmatrix}d_1\\3d_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix} = \begin{bmatrix}1\\-3\end{bmatrix}$

$\Longrightarrow d_1 = 1$ & $d_2 = -1$

Solution:

$$u = \tfrac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\cos t + 1\cdot\begin{bmatrix}1\\1\end{bmatrix}\sin t - \tfrac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}\cos 3t - 1\cdot\begin{bmatrix}-1\\1\end{bmatrix}\sin 3t$$

$u_1(t)$ = top component

$= \tfrac{1}{2}\cos t + \sin t + \tfrac{1}{2}\cos 3t + \sin 3t$

$u_2(t)$ = bottom component

$= \tfrac{1}{2}\cos t + \sin t - \tfrac{1}{2}\cos 3t - \sin 3t$

EX: Solve the initial value problem.

$c_1 = 5 \qquad m_1 = 1 \qquad u(0) = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$

$c_2 = 4 \qquad m_2 = 1/5 \qquad$ with $\qquad u'(0) =$

Write the answer as $u_1(t)$ & $u_2(t)$.

$$K = \begin{bmatrix} 5+4 & -4 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -4 & 4 \end{bmatrix} \qquad M^{-1}K = \begin{bmatrix} 9 & -4 \\ -20 & 20 \end{bmatrix}$$

eigenval. $\quad \det \begin{bmatrix} 9-\lambda & -4 \\ -20 & 20-\lambda \end{bmatrix} = 0$

$$\lambda^2 - 29\lambda + 100 = 0$$

$$(\lambda - 4)(\lambda - 25) = 0 \qquad \underline{\underline{\lambda = 4,\ 25}}$$

eigenvect.

$\lambda = 4 \quad \begin{bmatrix} 9-4 & -4 \\ -20 & 20-4 \end{bmatrix} = L \begin{bmatrix} 5 & -4 \\ 0 & 0 \end{bmatrix}$

$\omega = 2$

$\hookrightarrow$ eigenvect $= \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\lambda = 25 \quad \begin{bmatrix} 9-25 & -4 \\ -20 & 20-25 \end{bmatrix} = L \begin{bmatrix} -16 & -4 \\ 0 & 0 \end{bmatrix}$

$\omega = 5$

$\hookrightarrow$ eigenvect $= \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

general solution.

$$u = c_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix} \cos(2t) + d_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix} \sin(2t) + c_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \cos(5t) + d_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \sin(5t)$$

plug in initial position.

$$\begin{bmatrix} 4 & -1 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = u(0) = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{16+5} \begin{bmatrix} 4 & 1 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

plug in initial velocity.

$$\begin{bmatrix} 4 & -1 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 2d_1 \\ 5d_2 \end{bmatrix} = u'(0) = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} 2d_1 \\ 5d_2 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 4 & 1 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow d_1 = 1 \ \& \ d_2 = 0$$

solution.

$$u = 1 \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} \cos(2t) + 1 \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} \sin(2t) + (-2) \begin{bmatrix} -1 \\ 4 \end{bmatrix} \cos(5t) + 0 \cdot \begin{bmatrix} -1 \\ 4 \end{bmatrix} \sin(2t)$$

$$u = \begin{bmatrix} 4\cos(2t) + 4\sin(2t) + 2\cos(5t) \\ 5\cos(2t) + 5\sin(2t) - 8\cos(5t) \end{bmatrix} \begin{matrix} \leftarrow u_1 \\ \leftarrow u_2 \end{matrix}$$

$$\boxed{\begin{aligned} u_1 &= 4\cos(2t) + 4\sin(2t) + 2\cos(5t) \\ u_2 &= 5\cos(2t) + 5\sin(2t) - 8\cos(5t) \end{aligned}}$$